of thin elastic shells. In: Problems of Hydrodynamics and Mechanics of Continuous Media. "Nauka", Moscow, 1969.
3. Visarion, V. and Stănescu, C., Calculul stărilor de tensiune în teoria placilor curbe. Bucuresti Acad. Republicii Socialiste. Romania, 1969.
4. Rabotnov, Iu. N., Some solutions of the membrane theory of shells. PMM Vol. 10, № 5, 6, 1946.
5. Gol'denveizer, A. L. and Zveriaev, E. M., State of stress in unconstrained shells of zero curvature. PMM Vol. 35, № 2, 1971.

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ASYMPTOTIC METHOD OF INVESTIGATION OF SHORT-WAVE OSCILLATIONS OF SHELLS

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Short-wave oscillations of shells located in a certain sufficiently narrow boundary region, are investigated. An asymptotic integration procedure is constructed, by analogy with the methods presented in papers [1, 2]. Attention is paid mainly to the natural oscillations of shells, but forced oscillations are also considered at the end of this paper. The region of the oscillations here investigated is arbitrarily divided in two parts: one low frequency and the other high frequency. The equation of the first approximation for high frequency oscillations is the simplest, therefore this equation is considered first of all and the asymptotic procedure of integration is constructed; afterwards this method is generalized for low frequency short-wave oscillations of shells.

1. In this paper the oscillations are considered to be short-wave, if they are defined by the equations of a rapidly varying state of stress. Moreover, the so-called quasitransverse oscillations are considered when in the equations the inertial terms relating to tangential displacements, are discarded. With these assumptions the equations are written in the following form (using here the notation from the monograph [3]):

$$h_1^2 \Delta^2 w - E^{-1} h^{-1} \Delta_1 c - \lambda^2 w = 0, \quad \lambda^2 = \rho E^{-1} \omega^2 \quad Eh \Delta_1 w + \Delta^2 c = 0, \quad h_1^2 = h^2 [3(1-\sigma^2)]^{-1}, \quad \Delta = B^{-1}\partial_\alpha (B\partial_\alpha) + A^{-1}\partial_\beta (A\partial_\beta), \quad \partial_\alpha = A^{-1}\partial / \partial\alpha, \quad \partial_\beta = B^{-1}\partial / \partial\beta$$
$$\Delta_1 = B^{-1}\partial_\alpha (BR_2^{-1}\partial_\alpha) + A^{-1}\partial_\beta (AR_1^{-1}\partial_\beta) \tag{1.1}$$

It is assumed that the system of coordinates for the middle surface is referred to the principal lines of curvatures and the boundary is represented as a smooth convex line without corners (convexity condition will be considered later); the middle surface must be sufficiently smooth. Oscillations with frequencies satisfying the inequality

$$\lambda \gg \max (R_1^{-1}, R_2^{-1}) \tag{1.2}$$

in the whole region will be considered as high frequency oscillations. In this case we neglect those terms of the system which contain the radii of curvatures. Hence, constructing the first approximation we obtain the equation

$$h_1^2 \Delta^2 w - \lambda^2 w = 0 \qquad (1.3)$$

We note that the applied simplifications are entirely valid and the approximate solution, if required, can be made more exact within the given asymptotic error, using the known procedure. Here only the first approximation will be constructed. The oscillations defined by Eq. (1.3), are denoted in [4] as the quasitransverse oscillations of a large variability; nontangential boundary conditions are used for them (tangential conditions on the boundary are not taken into account in the first approximation).

Let us consider three most widely used forms of nontangential boundary conditions: hinged support, rigid clamping and free edge. In the case of a hinged support on the boundary, the perpendicular sag of the shell and the moment are equal to zero. These conditions are reduced to the following ones:

$$w = \partial_n^2 w = 0 \qquad (1.4)$$

For rigid clamping

$$w = \partial_n w = 0 \qquad (1.5)$$

For a free edge

$$\partial_n^2 w + \sigma \partial_s^2 w = 0, \quad \partial_n [\partial_n^2 w + (2 - \sigma) \partial_s^2 w] = 0$$

Equation (1.3) can be expressed by two equations

$$\Delta w_1 - \lambda_{pq} w_1 = 0, \quad \Delta w_2 + \lambda_{pq} w_2 = 0, \quad \lambda_{pq} = \lambda h_1^{-1} \qquad (1.6)$$

It is not difficult to see that in the case of a hinged support, it is sufficient to integrate the second equation for the boundary condition $w = 0$. The remaining second condition of (1.4) is satisfied automatically in conformity with the equation. The problem (1.5), as it will be shown later, is also reduced to the integration of the second order equation, i.e. of the second equation of (1.6). It gives the oscillating integrals and we shall proceed with their construction. We shall consider the oscillations located in a sufficiently narrow boundary region of the shell. We express the equation in the coordinates (s, n), where s is the length of the boundary arc, n is the arc length of the middle surface curve, orthogonal to the boundary

$$\Delta w + \lambda_{pq} c(n, s) w = 0, \quad \Delta = a(s, n) \partial_s^2 + b(s, n) \partial_n^2 \qquad (1.7)$$

Here only the terms with higher derivatives are retained and for purpose of generalization the coefficient c is added which does not complicate the mathematical operations. By introducing this coefficient, we facilitate a further generalization of the method for examination of low frequencies and, in addition, it can characterize a variable density or rigidity of the shell.

In conformity with [2] we construct the solution of Eq. (1.7) in the form

$$w = v(p^{2/3} \Psi) e^{ip\Phi} \qquad (1.8)$$

where Φ and Ψ are the unknown functions, p is the unknown larger parameter determined by the natural frequency (it is specified in the problem of forced oscillations), and v is the Airy function satisfying the equation

$$v''(x) - xv = 0$$

It will be necessary to use here the following asymptotic properties of the Airy function

$$v(-x \gg 1) = x^{-1/4} \sin\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right)(1 + O(x^{-3/2}))$$
$$v(x \gg 1) = {}^1/_2\, x^{-1/4} \exp(-{}^2/_3\, x^{3/2})(1 + O(x^{-3/2}))$$

We note that in [2] the Helmholtz equation has been considered for the region bounded by a convex curve. The integration procedure was constructed assuming that the caustics from the Helmholtz equation in [2] are known (caustics are the lines which have the following property: a ray leaving the caustic tangentially and reflected from the boundary according to the law of equality of the incident and reflected angles, returns tangentially to the caustic). The method suggested in this paper does not assume that the caustics are already known, they are found during the solution procedure; moreover, the method is generalized for equations with variable coefficients.

We substitute (1.8) into (1.7) and reducing the exponential factor we obtain

$$p^2 [\Psi (\nabla\Psi)^2 - (\nabla \Phi)^2]\, v - 2ip^{5/3} v' (\nabla\Psi\nabla\Phi) + ipv\, \Delta\Phi +$$
$$p^{2/3} v'\, \Delta\Psi + \lambda_{pq}\, cv = 0$$

$$\nabla = \sqrt{a}\, \mathbf{s}\, \partial_s + \sqrt{b}\, \mathbf{n}\, \partial_n, \quad (\nabla\Psi, \nabla\Phi) = a\Psi_{,s}\Phi_{,s} + b\Psi_{,n}\Phi_{,n}$$

Here λ_{pq} are the natural frequencies. Taking into account that the functions v, v' are linearly independent, we obtain

$$p^2 [\Psi (\nabla\Psi)^2 - (\nabla\Phi)^2] + ip\, \Delta\Phi + \lambda_{pq}\, c = 0$$
$$2ip^{5/3} (\nabla\Psi, \nabla\Phi) + p^{2/3}\, \Delta\Psi = 0$$

We find the functions Φ, ψ and the natural frequencies λ_{pq} in the form

$$\Phi = \sum\nolimits_0 \Phi_i p^{-i}, \quad \Psi = \sum\nolimits_0 \Psi_i p^{-i}, \quad \lambda_{pq} = p^2 + \varkappa_1 p + \sum\nolimits_0 p^{-i}\varkappa_{-i} \qquad (1.9)$$

Substituting these sums into the preceding equations and equating coefficients of equal powers of p, we obtain a recurrent system of equations

$$\Psi_0 (\nabla\Psi_0)^2 - (\nabla\Phi_0)^2 + c = 0, \quad (\nabla\Psi_0, \nabla\Phi_0) = 0 \qquad (1.10)$$
$$\Psi_i (\nabla\Psi_0)^2 + 2\Psi_0 (\nabla\Psi_0, \nabla\Psi_i) - 2(\nabla\Phi_0, \nabla\Phi_i) + \varkappa_{-i} = F_i$$
$$(\nabla\Phi_0, \nabla\Psi_i) + (\nabla\Psi_0, \nabla\Phi_i) = G_i$$

Here only the first pair of equations defining the functions Ψ_0 and Φ_0 is nonlinear. The right-hand sides of the following pairs of equations are known functions if the solutions of the preceding equations are known. The boundary conditions for the functions Φ and Ψ follow from the conditions for the function w. In the case of the problem (1.4) we have

$$p^{2/3}\, \Psi(s, n = 0) = t_q, \quad q = 0, 1, 2, \ldots, \quad [p\Phi] = 2n\pi, \quad n \gg 1$$

where t_q are the roots of the Airy function; the square brackets denote the increase of the function as a result of passing along the boundary of the region. Replacing the functions Ψ and Φ by their expansions, we obtain

$$\Psi_0 = t_q p^{-2/3}, \Psi_i = 0, i = 1, 2, 3, \ldots, [\Phi_0] = 2\, n\pi p^{-1}, [\Phi_i] = 0$$

In the problem (1.4) the function Ψ has a constant value on the boundary. This facilitates essentially the construction of the functions Ψ_i and Φ_i. The boundary conditions for all approximations, beginning with $i = 1$, are homogeneous.

Let us consider the conditions for a rigid clamping (1.5). We construct the solution for the system (1.6) transformed to coordinates (n, s) in the form

$$w_1 = Fe^{pf}, \quad w_2 = v\,(p^{2/3}\,\Psi)\, e^{ip\Phi} \tag{1.11}$$

The equations for the functions Ψ and Φ are written above. The equations for the first approximation of the functions f and F, if they are sought in the form of expansions (1.9) are the following:

$$(\nabla f_0)^2 - c = 0, \quad (\nabla f_0, \quad \nabla F_0) + F_0\,\Delta f_0 = 0$$
$$f = \Sigma f_i p^{-i}, \qquad F = \Sigma F_i p^{-i}$$

Substituting (1.11) into (1.5), we obtain on the boundary the equalities (postulating temporarily that $v\,(p^{2/3}\,\Psi)$ is a slowly varying boundary function)

$$f_0 = i\Phi_0, \quad f_1 = i\Phi_1, \quad v\,(p^{2/3}\,\Psi) + F = 0 \tag{1.12}$$
$$ip\Phi_{,n} + p^{2/3}\,\Psi_{,n}v' + pf_{,n}\,F + F_{,n} = 0$$

The boundary condition defines two functions of f. We choose the one for which the function w rapidly decreases with distance from the boundary to the region's interior (the rapidly increasing component of the solution is discarded). The boundary value of the function Ψ is small because the width of the boundary region is small. Then according to the second equation of (1.10) the product $\Psi_{,n}\ \Phi_{,n}$ must be small. The assumption that the derivative $\Psi_{,n}$ is small must be discarded since the known solution for the circle with $a = b = c = 1$ does not result from the general solution. Essentially, the suggested method is based on the fact that the form of the solution is guessed by comparing with the standard solution for the circle. As a result, we obtain that the function $\Phi_{,n}$ represents a small value on the boundary and can be neglected in the third equation of (1.12). Now equating for the boundary conditions (1.12) coefficients of the same powers of p, we obtain the following equality $v\,(p^{2/3}\,\Psi) = 0$; hence, in this case the boundary value of function Ψ is also equal to the value $\Psi = t_q\,p^{-2/3}$. The assumption that the derivative $\Phi_{,n}$ is small on the boundary now becomes a fact that on the boundary this derivative is equal to zero.

Analogously, it is proved that in the case of a free edge the first approximation of the function Ψ is equal to $\Psi = t_q' p^{-2/3}$, where t_q' are the roots of the function v'. The next approximation of the boundary values of the unknown functions are obtained by a successive writing-out of the boundary conditions in a recurrent system of conditions and equating the coefficients of the same powers of p.

2. We construct the solution of the first two equations from (1.10) (for simplicity the indices of the zero-order approximation are omitted)

$$\begin{gathered}\Psi\,(a\Psi_{,s}^2 + b\Psi_{,n}^2) - (a\Phi_{,s}^2 + b\Phi_{,n}^2) + c = 0 \\ a\Psi_{,s}\Phi_{,s} + b\Psi_{,n}\Phi_{,n} = 0 \\ \Psi\,(s, n = 0) = \varepsilon, \qquad [\Phi] = 2\,n\pi p^{-1}\end{gathered} \tag{2.1}$$

Here one of the two values $t_q p^{-2/3}$, $t_q' p^{-2/3}$ is denoted as a small value of ε. As the solution has to be found for a narrow strip $n < 1$, we construct the functions Ψ and Φ in the form of the Taylor expansions with respect to the coordinate n

$$\Psi = \sum_0 \Psi_i n^i, \qquad \Phi = \sum_0 \Phi_i n^i \tag{2.2}$$

where Ψ_i and Φ_i are functions of the boundary arc.

It follows from the first boundary condition and the second equation of (2.1), that

$$\Psi_0 = \varepsilon, \quad \Phi_1 = 0, \quad \Psi_{0,s} = 0$$

Substituting (2.2) into (2.1), after expanding the coefficients into series with respect to n and equating the coefficients of the same powers of n, we obtain a recurrent sequence of equations. We write here the equations determining the approximations Ψ_i, Φ_i, $i = 0, 1, 2$

$$\varepsilon b_0 \Psi_1^2 - a_0 \Phi_{0,s}^2 + c_0 = 0 \tag{2.3}$$

$$\varepsilon (4b_0 \Psi_1 \Psi_2 + b_1 \Psi_1^2) + b_0 \Psi_1^3 - a_1 \Phi_{0,s}^2 + c_1 = 0$$

$$a_0 \Psi_{1,s} \Phi_{0,s} + 2b_0 \Psi_1 \Phi_2 = 0$$

$$\varepsilon (a_0 \Psi_{1,s}^2 + 4b_0 \Psi_2^2 + 6b_0 \Psi_1 \Psi_3 + 4b_1 \Psi_1 \Psi_2 + b_2 \Psi_1^2) + 5b_0 \Psi_1^2 \Psi_2 + b_1 \Psi_1^3 - 2a_0 \Phi_{0,s} \Phi_{2,s} - a_2 \Phi_{0,s}^2 - 4b_0 \Phi_2^2 + c_2 = 0$$

$$a_0 \Psi_{2,s} \Phi_{0,s} + a_1 \Psi_{1,s} \Phi_{0,s} + 3b_0 \Psi_1 \Psi_3 + 4b_0 \Psi_2 \Phi_2 + 2b_1 \Psi_1 \Phi_2 = 0$$

$$a = \sum_0 a_i n^i, \quad b = \sum_0 b_i n^i, \quad c = \sum_0 c_i n^i$$

There is one arbitrary function ($\Phi_{0,s}$ or Ψ_1) in the system, the other functions are determined by it. As a small parameter $\Psi_0 = \varepsilon$ is present in the equations, we expand the required solutions in the asymptotic sums

$$\Phi_i = \sum_{j=0} \Phi_{ij} \varepsilon^j, \qquad \Psi_i = \sum_{j=0} \Psi_{ij} \varepsilon^j \tag{2.4}$$

Substituting these sums into system (2.3) and equating coefficients of the same powers of ε, we obtain a new recurrent system of equations. We present the equations for the first three approximations Φ_{i0}, Ψ_{i0}, $i = 0, 1, 2$

$$-a_0 \Phi_{00,s}^2 + c_0 = 0, \quad b_0 \Psi_{10}^3 - a_1 \Phi_{00,s}^2 + c_1 = 0$$

$$2b_0 \Psi_{10} \Phi_{20} + a_0 \Psi_{10,s} \Phi_{00,s} = 0$$

$$5b_0 \Psi_{10} \Psi_{20} + b_1 \Psi_{10}^3 - 2a_0 \Phi_{00,s} \Phi_{20,s} - a_2 \Phi_{00,s}^2 - 4b_0 \Phi_{20}^2 + c_2 = 0$$

$$3b_0 \Psi_{10} \Phi_{30} + a_0 \Psi_{20,s} \Phi_{00,s} + a_1 \Psi_{10,s} \Phi_{00,s} + 4b_0 \Psi_{20} \Phi_{20} + 2b_1 \Psi_{10} \Phi_{20} = 0$$

The equations for the second approximations Φ_{i1}, Ψ_{i1}, $i = 0, 1$

$$b_0 \Psi_{10}^2 - 2a_0 \Phi_{00,s} \Phi_{01,s} = 0$$

$$4b_0 \Psi_{10} \Psi_{20} + b_1 \Psi_{10}^2 + 3b_0 \Psi_{10} \Psi_{11} - 2a_1 \Phi_{00,s} \Phi_{01,s} = 0$$

$$a_0 (\Psi_{10,s} \Phi_{01,s} + \Psi_{11,s} \Phi_{00,s}) + 2b_0 (\Psi_{10} \Phi_{21} + \Psi_{11} \Phi_{20}) = 0$$

The equations for the third approximations Ψ_{12}, Φ_{02}

$$- a_0\Phi_{00,s}\Phi_{02,s} - a_0\Phi_{01,s}^2 + b_0\Psi_{10}\Psi_{20} = 0$$

$$3b_0\Psi_{10}^2\Psi_{12} + 3b_0\Psi_{10}\Psi_{11}^2 + 4b_0(\Psi_{10}\Psi_{21} + \Psi_{11}\Psi_{20}) + 2b_1\Psi_{10}\Psi_{11} - a_1(\Phi_{01,s}^2 + 2\Phi_{00,s}^2\Phi_{02,s}) = 0$$

We write the following approximations:

$$\Phi_{00} = \pm\int_{s_0}^{s} \sqrt{\frac{c_0}{a_0}}\, ds, \quad \Psi_{10} = \left[\frac{1}{b_0}\left(a_1\frac{c_0}{b_0} - c_1\right)\right]^{1/3}$$

$$\Phi_{01} = \int_{s_0}^{s} \frac{b_0\Psi_{10}}{2a_0\Phi_{00,s}}\, ds$$

Substituting the function Φ into the second boundary condition (2.1), we obtain the equation for determining the parameter p and the first approximation for the natural frequency

$$p = 2n\pi\left(\int_{\Gamma} \sqrt{\frac{c_0}{a_0}}\, ds\right)^{-1}$$

To construct the natural frequency with an acceptable accuracy when in the neighborhood there is no other natural frequency within the range of error of determination, it is necessary to construct the second approximation for the expansion (1.9), i.e. to find $\varkappa_1$ from the condition $[\Phi_1] = 0$. In this case λ_{pq}, the error of determination is of the order $O(p^{-1})$, i.e. essentially smaller than the average interval between the eigenfunctions which is of the order of $O(1)$. To construct the further approximations to the natural frequencies of oscillation of shells, the equations quoted here are not sufficient, and more elaborate equations have to be used. Therefore we limit ourselves to the first approximation only, i.e. we define the frequencies with an error of $O(1)$.

We note that the oscillations located in the boundary zone are not possible in every region of the shell. It is not difficult to show that the condition

$$b_0^{-1}\left(a_1\frac{c_0}{b_0} - c_1\right) > 0 \qquad (2.5)$$

must be satisfied on the boundary.

If this inequality is not satisfied, the eigenfunctions cannot be constructed in the form (1.8). This condition is a generalized condition of convexity of the boundary curve in the problem for the Helmholtz equation on a plane [2]. Let us consider the line on which the argument of the Airy function vanishes, $\Psi = 0$. This line separates the region of the shell, where the solution rapidly oscillates $(\Psi < 0)$ from the region $(\Psi > 0)$ in which the eigenfunctions rapidly decrease with the distance from the line. This line belongs to the family of so-called caustics. If we limit the expansion (2.2) to two terms, the caustic is defined by the equation

$$n = \Psi_0\Psi_1^{-1}, \quad \Psi_0 = \varepsilon, \quad \Psi_1 = \Psi_{10} + \varepsilon\Psi_{11}$$

Let us consider some properties of the functions Φ, Ψ in the caustic's neighborhood. We transform Eq. (2.1) into the coordinate system (τ, ν), where τ is the length of the caustic arc, ν is the distance along its normal. It follows from the equations that

$$\Psi = 0, \; \Phi_{,\tau}^2 = ca^{-1}, \; \Phi_{,\nu} = 0$$

Differentiating the first equation of (2.1) with respect to ν and substituting the values of the functions obtained, we find that on the caustic

$$\Psi^{\prime 3}_{,\nu} = \frac{1}{b}\left(a_{,n}\frac{c}{a} - c_{,n}\right) = \frac{1}{b_0}\left(a_1\frac{c_0}{a_0} - c_1\right)$$

where a_i, b_i, c_i, $i = 0,1$ are the coefficients of the functions a, b, c expanded in Taylor series with respect to ν in the neighborhood of the caustic. The limitation (2.5) concerning the form of the boundary is also valid for the caustic.

Introducing a system of coordinates formed by the family of τ-lines, where $\Psi = \text{const}$, and the family of ν-lines orthogonal to τ-lines, the solution of the system (2.1) obtains the form

$$\Phi = \int_{\tau_0}^{\tau} \sqrt{\frac{c_0(\tau)}{a_0(\tau)}}\, d\tau, \qquad \Psi^{\prime 3/2} = \frac{3}{2}\int_0^{\nu} b^{-1/2}\left(a_{,\nu}\frac{c}{a} - c_{,\nu}\right)^{1/2} d\nu$$

Formally, a separation of the coordinates takes place in this system, the function Φ depends only on τ, while Ψ depends only on ν.

Let us show the relation between the system (2.1) and the eikonal equation. If we introduce the substitution

$$S^+ = \Phi - {}^2/_3\,(-\Psi)^{3/2}, \qquad S^- = \Phi + {}^2/_3\,(-\Psi)^{3/2} \tag{2.6}$$

the system (2.1) is reduced to the equations

$$(\nabla S^{\pm})^2 = c \tag{2.7}$$

The problem would be reduced to these equations (really to one equation), if the solution was constructed not in the form (1.8) but in the form

$$w = Fe^{ipf}$$

3. Now let us consider the range of medium frequencies quasi-transverse oscillations of great variability when the frequencies are of the order of maximum of the principal curvatures of the middle surface, but exceeding this maximum. In this case the state of stress is defined by Eqs. (1.1). Limiting the examination to the first approximation, we can neglect in these equations the terms with lower derivatives. Then, the system is reduced to one equation

$$h_1^2\Delta^4 w + \Delta_1^2 w - \lambda^2\Delta^2 w = 0$$

We construct the solution in the form (2.10)

$$w = Fe^{ipf}, \quad \lambda^2 = \lambda_0 + p^{-1}\lambda_1 + \ldots, \quad p = h_1^{-1/2} \tag{3.1}$$

As a matter of fact, variability of the solution of Eqs. (1.1) is specified, i.e. the parameter p is known in this case. The asymptotic procedure of integration for the solution in the form (3.1) is presented in [5–7]. Unlike in these papers we have here oscillating integrals, i.e. there are purely imaginary solutions for the function f.

As before, we shall examine the oscillations located in a sufficiently narrow boundary zone. We transform the equation to coordinates (s, n), assuming for simplicity that the boundary coincides with the line of principal curvature. The first approximation of the function f satisfies the following equation (we omit the index zero):

$$(\nabla f)^8 + (\nabla_1 f)^4 - \lambda_0 (\nabla f)^4 = 0 \tag{3.2}$$

$$(\nabla f)^2 = af_{,s}^2 + bf_{,n}^2, \quad (\nabla_1 f)^2 = a^\circ f_{,s}^2 + b^\circ f_{,n}^2$$

$$a = A^{-2}, \quad b = B^{-2}, \quad a^\circ = A^{-2}R_2^{-1}, \quad b^\circ = B^{-2}R_1^{-1}$$

It was assumed in [5 – 7] that the function f on the boundary is specified. In the eigenvalue problem the boundary value is not specified, it must be determined. Equation (3.2) has purely imaginary solutions which in this case are the fundamental ones and have to be constructed first. The remaining real and complex integrals are defined by the boundary values of the imaginary solution, as the function f (1.12) was determined in Sect. 2.

We construct the imaginary integrals of Eq. (3.2) using the substitution (2.6) which allowed Eq. (2.7) to be replaced by the system (2.1), which is convenient for constructing the solution in the form of series

$$f^{\pm} = \Phi \mp {}^2/_3 (-\Psi)^{3/2} \tag{3.3}$$

Substituting (3.3) into (3.2) we obtain the following two equations for the functions Φ and Ψ

$$L_1^4 + 6L_1^2L_2^2 + L_2^4 + L_1^{\circ 2} + L_2^{\circ 2} - \lambda_0 (L_1^2 + L_2^2) = 0$$

$$2L_1L_2 (L_1^2 + L_2^2) + L_1^{\circ} L_2^{\circ} - \lambda_0 L_1 L_2 = 0$$

$$L_1 = (\nabla\Phi)^2 - \Psi (\nabla\Psi)^2, \quad L_2 = 2 (-\Psi)^{1/2} (\nabla\Psi, \nabla\Phi)$$

The functionals L_1°, L_2° are obtained replacing the operator ∇ by ∇_1 in the functionals L_1, L_2. The boundary conditions for the functions Φ and Ψ have the form (2.1), i.e. they are like those in Sect. 2 (it is obvious that the derivation of the conditions (2.1) is generalized for the problem considered here). As previously, the value of the function Ψ is small on the boundary.

We find the functions Φ and Ψ in series form (2.2), the coefficients Φ_i and Ψ_i of which we express by the asymptotic sums (2.4). Substituting, as described above, the series into the equations and equating the coefficients of like powers of n and of the parameter p, we obtain a recurrent system of equations. We cite the equations only for the first approximation (the other equations are not given here as they are somewhat cumbersome; they have the structure of the equations presented in Sect. 2):

$$\Psi_0 = \varepsilon, \quad \Phi_1 = 0, \; a_0^4\Phi^8_{00,s} + (a_0^{\circ 2} - \lambda_0 a_0^2)\Phi^4_{00,s} = 0$$

$$2a_0^3 a_1 \Phi^6_{00,s} + (2b_0 a_0^3 \Phi^4_{00,s} + b_0^{\circ} a_0 \Phi^2_{00,s} - \lambda_0 b_0 a_0)\Psi_{10}^3 + a_0^{\circ} a_1^{\circ} \Phi^2_{00,s} - \lambda_0 a_0 a_1 \Phi^2_{00,s} = 0$$

Discarding the zero roots in the equation for $\Phi_{00,s}$, we obtain

$$a_0^4\Phi^4_{00,s} = \lambda_0 a_0^2 - a_0^{\circ 2} \tag{3.4}$$

It has been assumed above that the frequencies under consideration exceed in value the principal curvatures of the middle surface in the whole region. In this case the right side of Eq. (3.4) exceeds zero and consequently, there are two pure imaginary solutions

$$a_0^2\Phi^2_{00,s} = -(\lambda_0 a_0^2 - a_0^{\circ})^{1/2} \tag{3.5}$$

We note that it is possible to reduce somewhat the constraint on the values of the frequencies and to require that they exceed the principal curvatures only in a certain boundary region which exceeds dimensionally the oscillation zone of the eigenfunctions. The limitation (1.2) can also be relaxed in that the inequality has to be satisfied in the neighborhood of the boundary. The equation of the first approximation Ψ_{10} of the function Ψ can be represented in the form

$$\rho\Psi_{10}{}^3 = \gamma, \quad \rho = 2b_0a_0{}^3\Phi_{00,s}^4 + b_0{}^\circ a_0\Phi_{00,s}^2 - \lambda_0 b_0 a_0$$

$$\gamma = 2a_0{}^3a_1\Phi_{00,s}^6 + (a_0{}^\circ a_1{}^\circ - \lambda_0 a_0 a_1)\Phi_{00,s}^2$$

Hence, it is obvious that the coefficients of Eq. (1.1) and the boundary must be such that the function Ψ_{10} differs from zero $(\gamma\rho^{-1} > 0)$. This condition represents a generalization of the condition (2.5). Only by satisfying this inequality, the iteration procedure for constructing the following approximations is possible and convergent.

The first approximation for the natural frequencies is determined from the periodicity condition of the function w with respect to the boundary

$$p\int_\Gamma \Phi_{00,s}\,ds = 2n\pi$$

Substituting the value of p from (3.1) and $\Phi_{00,s}$ from (3.5), we obtain the following equation for λ_0:

$$\int_\Gamma a_0^{-1}(\lambda_0 a_0{}^2 - a_0{}^{0\,2})^{1/4}ds = 2n\pi h_1^{1/4}$$

As in Sect. 2, the first approximation for the natural frequency does not depend on the exact form of the boundary conditions, the influence of which appears in further approximations.

The boundary of the oscillating zone is defined by the equation $\Psi = 0$, the first approximation of which has the form: $n = -\Psi_0\Psi_{10}{}^{-1}$.

Beside the oscillating part of the solution, three other solutions of the form (3.1), rapidly decreasing with the distance from the boundary, take part in satisfying the boundary conditions. The functions f in the index of the exponent, are constructed from Eq. (3.2) with respect to the boundary value $f_k = i\Phi$, where Φ is defined by the formula (3.5). These functions participate in the iteration procedure, starting with the second step. We note that the question of constructing the function f is considered in detail in [6, 7].

The problem of forced oscillations of shells with rapidly oscillating boundary conditions is solved in the following manner. An integration method for the differential equations with rapidly oscillating boundary conditions was worked out in [6, 7] for the case when all integrals (3.1) decrease with the distance from the boundary. This method is applied to the problem of forced oscillations with the following alterations. According to the method suggested above, the functions f which are pure imaginary, are found for specified boundary values. The integration procedure shows whether the given frequency coincides with the natural one. If there is no coincidence, the integration is performed within the required asymptotic accuracy.

REFERENCES

1. Babich, V. M. and Buldyrev, V. S., Asymptotic methods in the problem of diffraction of short waves. Moscow, "Nauka", 1972.
2. Lazutkin, V. F., Asymptotics of the eigennumbers of the Laplace operator and quasi-modes. Quasi-mode series corresponding to the system of caustics near the boundary region. Izv. Akad. Nauk SSSR, Ser. matem., Vol. 37, № 2, 1973.
3. Gol'denveizer, A. L., Theory of elastic thin shells. (English translation), Pergamon Press, Book № 09561, 1961.
4. Gol'denveizer, A. L., Classification of integrals of the dynamic equations

of the linear two-dimensional theory of shells. PMM Vol. 37, № 4, 1973.
5. Vishik, M. I. and Liusternik, L. A., On the asymptotics in the solution of problems with rapidly oscillating boundary conditions for the equations with partial derivatives. Dokl. Akad. Nauk SSSR, Vol. 119, № 4, 1958.
6. Gol'denveizer, A. L., Asymptotic integration of partial differential equations with boundary conditions depending on one parameter. PMM Vol. 22, № 5, 1958.
7. Gol'denveizer, A. L., Asymptotic integration of linear partial differential equations with a small principal part. PMM Vol. 23, № 1, 1959.

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FRACTURE MECHANICS OF PIEZOELECTRIC MATERIALS. AXISYMMETRIC CRACK ON THE BOUNDARY WITH A CONDUCTOR

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A condition is formulated which is the generalization of the fracture variational principle in piezoelectric media. In some cases such a representation of the fracture condition, which permits the determination of crack development in a piezoelectric material, turns out to be preferable to the analogous condition obtained in [1].

The problem of a disc-shaped crack developing on the boundary between a piezoelectric ceramic and an elastic isotropic conductor is considered as an illustration.

1. Variational principle of the fracture mechanics of piezoelectric media. The stress components σ_{ij} $(i, j = 1, 2, 3)$ and the components of the electric induction vector of a piezoelectric medium satisfy the equilibrium equations and the Maxwell equation in the statistical case

$$\frac{\partial\sigma_{ij}}{\partial x_j} = 0, \qquad \frac{\partial D_j}{\partial x_j} = 0 \tag{1.1}$$

In Cartesian coordinates referred to the crystal-physics axes, for a piezoelectric medium [2]

$$\sigma_{ij} = c_{ijkl}^E\,\varepsilon_{kl} - e_{ijk}\,E_k \tag{1.2}$$

$$D_i = e_{kli}\varepsilon_{kl} + \varepsilon_{ik}{}^S E_k \quad (i, j, k, l = 1, 2, 3)$$

Here c_{ijkl}^E are the elastic moduli of the medium, e_{ijk} are the piezoelectric moduli, $\varepsilon_{ik}{}^S$ are the adiabatic dielectric constants, E_k are the electric field strength components, and ε_{kl} are the strain tensor components.

To derive the condition governing crack development in a piezoelectric material, let us examine a number of possible body states just as in [3, 4]. Suppose there is no crack in the body in State 1, and external loads and an electrical potential φ $(E_k = \partial\varphi/\partial x_k)$ is specified on the body surface S. The stresses σ_{ij1}, the displacement vector u_{i1}, the